This post is the first in what should be a fairly long sequence of articles concerning fluid mechanics. The aim of this series is to present an intuitive feel for the phenomena of fluid mechanics from a physical point-of-view and a clean way of understanding the mathematical models used to describe them. This last point deserves some comments especially.

One need only consult the numerous textbooks that abound in the marketplace of ideas to walk away with a profound sense of bewilderment at all the possible ways to express the fundamental, governing equations of fluid mechanics. One’s confusion further increases by both a lack of standardization of terminology and by the failure to keep a sharp line between essential principles, such as the material derivative or the conservation of energy, and those relationships tailored for a specific application, such as the connection between applied stress and the strain rate for a Newtonian fluid.

This post attempts to improve this situation by deriving the governing equations, with no simplifications or specializations, using a minimalist approach consisting of the following ingredients

1. The definition of the material derivative (see… for details) to move between the particle-flow (Lagrangian) and field (Euler) points-of-view (see… for details)
2. The divergence theorem
3. The Reynolds transport theorem connecting the time rate-of-change of a volume of fluid to its flow velocity (see… for details)
4. Surface forces defined in terms of the stress tensor
5. The conservation of mass
6. Newton’s second law
7. The First Law of Thermodynamics

This presentation reflects the influence of many sources, but the primary ones are *Mechanics* by Symon, *The Governing Equations of Fluid Mechanics* by Anderson found in Chapter 2 of *Computational Fluid Mechanics*, edited by Wendt, and *An Introduction to Fluid Mechanics* by Batchelor. All of these are fine resources but each has a particular drawback that this synthesis is designed to address.

The governing equations of fluid mechanics in the particle-flow picture are particularly simple. In this picture, we imagine looking as a small fluid element surrounded by either other fluid elements or boundaries with other materials. By tracking the interactions of this element with its environment, one can easily write down the governing equations in familiar terms. Transformation to the field picture is then facilitated by applying a combination of the items 1-3 in the list above.

Broadly speaking there are two ways to combine items 1-3 to arrive at the field picture: the ordinary and flux-conservative forms (with many variations in each). As discussed in detail in Anderson, the flux-conservative form, written generally as

\[ \frac{\partial \alpha}{\partial t} + \nabla \cdot \vec j\_{\alpha} = 0 \; , \]

for a physical quantity $$\alpha$$ and corresponding current $$\vec j\_{\alpha}$$ is the preferred form as it leads to better behaved equations, especially when dealing with shocks. The ‘fork in the road’ between these two ways will be obvious below and will center on either explicitly using mass conservation in the equations (ordinary) or ignoring it (flux-conservative).

## Conservation of Mass = Continuity Equation

The conservation of mass in the particle-flow picture is simply

\[ \frac{d}{dt} m = 0 \; ,\]

which states that, no matter how the boundary of the element deforms as it moves, the amount matter contained within stays constant.

Defining the particle mass in the usual way, $$m = \rho {\mathcal Vol}$$, in terms of the mass density and the volume, gives

\[ \frac{d}{dt} \left( \rho {\mathcal Vol} \right) = \frac{d \rho}{dt} {\mathcal Vol} + \rho \frac{d}{dt} {\mathcal Vol} \; . \]

Next using Reynold’s transport theorem to eliminate the time derivative of the volume gives

\[ \frac{d \rho}{dt} {\mathcal Vol} + \rho \nabla \cdot {\vec V} \, {\mathcal Vol} = 0 \; , \]

The next step consists of using the form of the material derivative in the field picture to arrive at

\[ \left( \frac{\partial \rho}{\partial t} + {\vec V}\cdot \nabla \rho \right) {\mathcal Vol} + \rho \nabla \cdot {\vec V} \, {\mathcal Vol} = 0 \; .\]

Combining the second the third term and then dividing out the $${\mathcal Vol}$$ gives the flux-conservative form of the continuity equation

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot \left( \rho {\vec V} \right) = 0 \; . \]

## Newton’s Second Law = Cauchy Momentum Equation

Newton’s second law in the particle-flow picture looks identical to how it is framed in elementary classical mechanics.

\[ \frac{d}{dt} \left( m \vec V \right) = \vec F\_{total} \; , \]

where the right-hand side is the sum is over all the forces exerted on the element.

Break the right-hand side into surface forces due to exchanges of momentum only at the boundary of the fluid element and body forces (e.g. gravity) where momentum exchange takes place between a field and every part of the element:

\[ \vec F\_{total} = \sum\_{faces} {\vec f}\_s + {\vec f}\_b {\mathcal Vol} \; .\]

Expressing the surface force on a given face in terms of the stress tensor as $${\vec f}\_s = {\mathbf T} \cdot {\hat n} \, {\mathcal Area} $$, simplifies the first term to

\[ \sum\_{faces} {\vec f}\_s = \sum\_{faces} {\mathbf T} \cdot {\hat n} \, {\mathcal Area} \; . \]

Applying the divergence theorem gives the final form of the resultant of the surface forces

\[ \sum\_{faces} {\vec f}\_s = \nabla \cdot {\mathbf T} \, {\mathcal Vol} \; .\]

Now examine the left-hand side first, which we will call $${\dot {\vec p }}$$.

\[ {\dot {\vec p }} = \frac{d}{dt} \left(m {\vec V} \right) = \frac{d m}{dt} {\vec V} + m \frac{d}{dt} {\vec V} \; . \]

There is a natural temptation to set the first term to zero and then to follow the same strategy as above. Suppose this is done then

\[ \rho \, {\mathcal Vol} \frac{d}{dt} {\vec V} = \nabla \cdot {\mathbf T} \, {\mathcal Vol} + {\vec f}\_b {\mathcal Vol} \; .\]

Dividing out the $${\mathcal Vol}$$ and expanding the material derivative gives

\[ \rho \left( \frac{\partial \vec V}{\partial t} + {\vec V} \cdot \nabla {\vec V} \right) = \nabla \cdot {\mathbf T} + {\vec f}\_b \; . \]

While this is a valid fluid equation, it is not in flux-conservative form and, as cited above, experience has shown that the best equations for computational modeling are in flux-conservative form. So we retain the $$(dm/dt) {\vec V}$$ and perform similar steps as to the above to get the left-hand side into the form

\[ \frac{d}{dt} {\vec p} = \frac{dm}{dt} {\vec V} + m \frac{d {\vec V}}{dt} = \left( \frac{\partial \rho}{\partial t} + \nabla \cdot {\rho {\vec V}} \right) {\mathcal Vol} + \rho \, {\mathcal Vol} \left( \frac{\partial {\vec V}}{\partial t} + {\vec V} \cdot \nabla {\vec V} \right) \; . \]

Factoring out $${\mathcal Vol}$$ and combining terms yields

\[ \frac{d}{dt} {\vec p} = {\mathcal Vol} \left( \frac{\partial (\rho {\vec V})}{\partial t} + \nabla \cdot ({\rho {\vec V}{\vec V}}) \right) \; . \]

Substituting in this form and dividing out the volume gives the Cauchy momentum equation in flux form

\[ \frac{\partial (\rho {\vec V})}{\partial t} + \nabla \cdot ({\rho {\vec V}{\vec V}}) = \nabla \cdot {\mathbf T} + {\vec f}\_b \; .\]

## First Law of Thermodynamics = The Energy Equation

The final equation is the first law of thermodynamics, which in the particle-flow picture looks like

\[ \frac{d}{dt} E = \frac{d {\mathcal W}}{dt} + \frac{ d{\mathcal Q}}{dt} \; , \]

Where $$E$$ is the total energy of the element, $$W$$ is the work done to the fluid element, and $$Q$$ is the heat that flows into it. As in treatment of the Cauchy momentum equation, we need to distinguish between work due to surface forces and work due to body forces. We must also distinguish between heat produced within the fluid element and heat that flows in from the boundary.

Because there are so many possible variations, we will content ourselves with one of the possible forms. Additional posts will explore other forms of this equation and additional assumptions.

To start, let’s look at the left-hand side. The total energy is the sum element’s kinetic energy and its internal potential energy

\[ E = \frac{1}{2} m {\vec V}^2 + U \; .\]

Defining the total energy through the relation $$E = m e\_t$$ with the specific total energy $$e\_t = 1/2 V^2 + u$$, we can express the left-had side as

\[ \frac{d}{dt} E = \frac{d}{dt} \left( m e\_t \right) = \frac{dm}{dt} e\_t + m \frac{d e\_t}{dt} \; .\]

As in the case of the momentum equation, keeping the time-derivative of mass explicitly around leads to the flux-conservative form. This is done with the usual application of the Reynold’s transport theorem and the definition of the material derivative to arrive at

\[ \left[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec V) \right] e\_t {\mathcal Vol} + \rho {\mathcal Vol} \left[ \frac{\partial e\_t}{\partial t} + {\vec V} \cdot \nabla (e\_t) \right] \; . \]

These two terms combine nicely to yield

\[ \left[ \frac{\partial \rho e\_t}{\partial t} + \nabla \cdot (\rho e\_t \vec V) \right] {\mathcal Vol} \; , \]

for the left-hand side.

Over on the right-hand side the time rate of work done is given by

\[ \frac{d W}{dt} = \vec F\_{total} \cdot \vec V = \vec f\_{body} \cdot \vec V {\mathcal Vol} + \sum\_{faces} \vec f\_{s} \cdot \vec V \; .\]

The last term can be put into a more useful form by again expressing the surface forces in terms of the stress tensor to get that the work done by these forces is then

\[ \sum\_{faces} \vec f\_s \cdot \vec V = \sum\_{faces} ({\mathbf T} \cdot \vec V ) \cdot {\hat n} \, {\mathcal Area} \; .\]

Finally apply the divergence theorem to get

\[ \sum\_{faces} \vec f\_s \cdot \vec V = \nabla \cdot ({\mathbf T} \cdot \vec V ) {\mathcal Vol} \; ,\]

which, when substituted back, gives a final form for the rate of work performed of

\[ \frac{d W}{dt} = \left[ \vec f\_{body} \cdot \vec V + \nabla \cdot ({\mathbf T} \cdot \vec V ) \right] {\mathcal Vol} \; .\]

The rate of change in the heat in the element is given by

\[ \frac{dQ}{dt} = \rho \dot q {\mathcal Vol} + \sum\_{faces} \vec h \cdot \hat n {\mathcal Area} \; , \]

where the first term accounts for the volumetric heating occurring within the element and the second term represents the heat flux $$\vec h$$ entering through the faces.

Applying the divergence theorem to the last term gives a final form for the rate of change of the heat of

\[ \frac{dQ}{dt} = \left[ \rho \dot q + \nabla \cdot \vec h \right] {\mathcal Vol}\; . \]

Equating all of these pieces and dividing out the volume yields the general form of the energy equation in flux-conservative form

\[ \frac{\partial \rho e\_t}{\partial t} + \nabla \cdot (\rho e\_t \vec V) = \vec f\_{body} \cdot \vec V + \nabla \cdot ({\mathbf T} \cdot \vec V ) + \rho \dot q + \nabla \cdot \vec h \; . \]

Next month’s columns will begin to look at specific physical solutions to this equations in simple settings.